Determining source cumulants in femtoscopy with Gram-Charlier and Edgeworth series

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Outline

Nomenclature

Measured correlation function

$$C(\mathbf{q}) = \frac{\rho(\mathbf{q})}{\rho^{ref}(\mathbf{q})}$$

Interpret as a probability density function

$$f(\mathbf{q}) = \frac{C(\mathbf{q}) - 1}{\int [C(\mathbf{q}) - 1] \, d\mathbf{q}}$$

Assume noninteracting, identical particles

$$C(\mathbf{q}) - 1 = \int d^3x S(\mathbf{x}) e^{i\mathbf{q}\cdot\mathbf{x}}$$

$$C(-\mathbf{q}) = C(\mathbf{q})$$

Objective: Describe the shape of the source $S(x)$ in terms of the shape of the correlator $C(q)$ using cumulants.

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$\kappa_r^{(q)}$ Known

$\kappa_r^{(x)}$ Unknown

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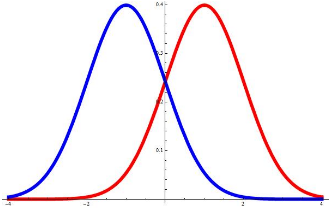
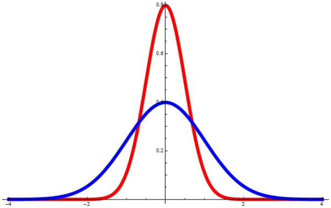
Gram-Charlier
Edgeworth

How cumulants describe shape

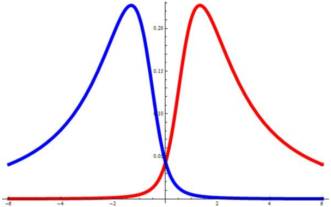
Mean $\kappa_1 < 0 < \kappa_1$



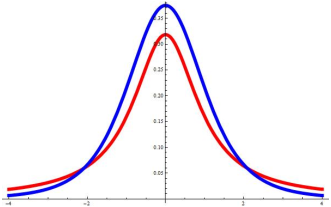
Variance $\kappa_2 < \kappa_2$



Skewness $\kappa_3 < 0 < \kappa_3$



Kurtosis $\kappa_4 < \kappa_4$



Useful properties of cumulants

- Translation invariant
- Additive under sums of random variables
- Simpler than moments, Gaussian cumulants:

$$\kappa_1 = \mu,\ \kappa_2 = \sigma^2,\ \kappa_r = 0 \text{ for all } r \geq 3$$

Shift and squeeze, $q \longrightarrow \frac{q-\mu}{\sigma}$
Yields standardised cumulants: $\gamma_1 = 0$, $\gamma_2 = 1$

$$\gamma_r = \frac{\kappa_r}{\sigma^r}$$

Guassian as a baseline, with $\gamma_4 \neq 0$ the distance from it

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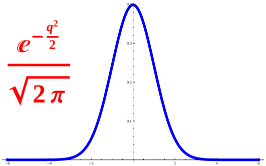
Measuring cumulants

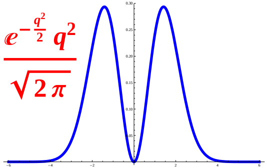
Defining moments

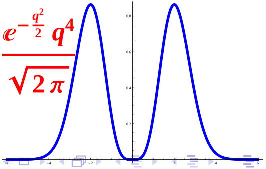
$$\mu_r^{(q)} = \int q^r f(q) dq$$

Cumulants in moments: $\kappa_r^{(q)}$

$$\kappa_2^{(q)} = \mu_2^{(q)}$$
$$\kappa_4^{(q)} = \mu_4^{(q)} - 3(\mu_2^{(q)})^2$$
$$\kappa_6^{(q)} = \mu_6^{(q)} - 15\mu_4^{(q)}\mu_2^{(q)} + 30(\mu_2^{(q)})^2$$







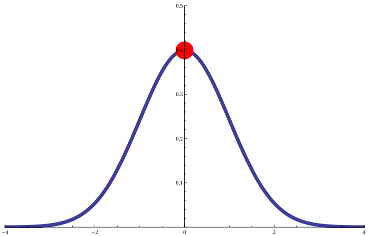
Symmetry of cumulants and generating functions

Generating function: q-cumulants

$$\kappa_r^{(q)} = (-i)^r \frac{d^r}{dx^r} \log S(x) \bigg|_{x=0}$$

Generating function: Source cumulants

$$\kappa_r^{(x)} = (-i)^r \frac{d^r}{dq^r} \log \frac{f(q)}{f(0)} \bigg|_{q=0}$$



- Easy to measure
- Hard to measure

Gram-Charlier expansion

Series expansion: Gram-Charlier series

$$f(q) = f_0(q) - c_1 f_0^{(1)}(q) + \frac{c_2}{2!} f_0^{(2)}(q) - \frac{c_3}{3!} f_0^{(3)}(q) + \cdots$$

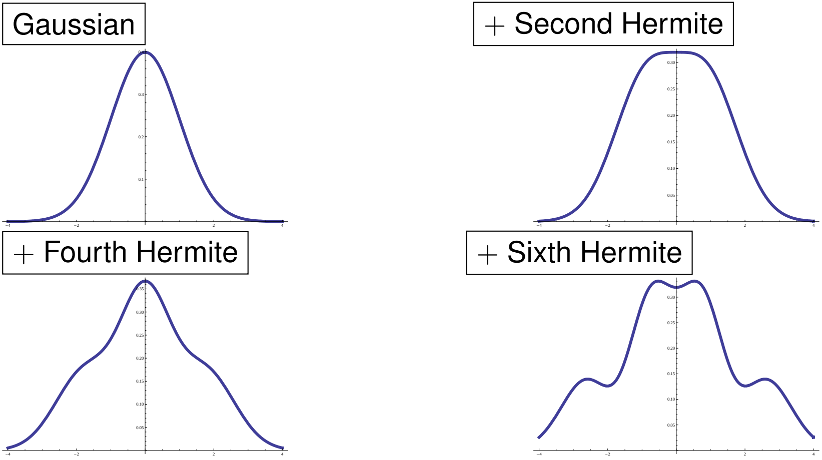
Reference function: Gaussian Distribution

$$f_0(q) = \frac{1}{\sqrt{2\pi}} e^{-\frac{q^2}{2}}$$

Derivatives: Hermite functions

$$\left(\frac{d}{dq}\right)^r \frac{1}{\sqrt{2\pi}} e^{-\frac{q^2}{2}} = H_r(q) \frac{1}{\sqrt{2\pi}} e^{-\frac{q^2}{2}}$$

Gram-Charlier in pictures



Relating coefficients to cumulants

Gram-Charlier with Gaussian reference function:

$$f(q) = f_0(q) - \frac{c_4}{4!} H_4(q) f_0(q) + \frac{c_6}{6!} H_6(q) f_0(q) + \cdots$$

Fourier Transform

$$\frac{S(x)}{f(0)} = e^{-\frac{x^2}{2}} \left[1 + \frac{c_4}{4!}(ix)^4 + \frac{c_6}{6!}(ix)^6 + \frac{c_8}{8!}(ix)^8 + \cdots \right]$$

Expand in cumulants

$$\frac{S(x)}{f(0)} = \exp\left[\frac{\kappa_2^{(q)}}{2!}(ix)^2 + \frac{\kappa_4^{(q)}}{4!}(ix)^4 + \frac{\kappa_6^{(q)}}{6!}(ix)^6 + \cdots \right]$$

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Coefficients in terms of cumulants

Equate series and compare powers in x^m

$$e^{-\frac{x^2}{2}}\left[1+\frac{c_4}{4!}(ix)^4+\frac{c_6}{6!}(ix)^6+\frac{c_8}{8!}(ix)^8+\cdots\right]$$
$$=e^{-\frac{x^2}{2}}\left[1+\frac{\kappa_4^{(q)}}{4!}(ix)^4+\frac{\kappa_6^{(q)}}{6!}(ix)^6+\frac{\kappa_8^{(q)}+35(\kappa_4^{(q)})^2}{8!}(ix)^8\cdots\right]$$

For symmetrical distributions

$$c_4=\kappa_4^{(q)}$$
$$c_6=\kappa_6^{(q)}$$
$$c_8=\kappa_8^{(q)}+35(\kappa_4^{(q)})^2$$
$$c_{10}=\kappa_{10}^{(q)}+210\kappa_6^{(q)}\kappa_4^{(q)}$$
$$c_{12}=\kappa_{12}^{(q)}+495\kappa_4^{(q)}\kappa_8^{(q)}+462(\kappa_6^{(q)})^2+5775(\kappa_4^{(q)})^3$$

Source cumulants in terms of q cumulants

Connecting cumulants

$$\kappa_2^{(x)} = \frac{1}{\kappa_r^{(q)}} \frac{f^{(2)}(q)}{f(q)}\bigg|_{q=0} \qquad \left\{ \kappa_2^{(x)} = (-i)^r \frac{d^r}{dq^r} \log \frac{f(q)}{f(0)}\bigg|_{q=0} \right\}$$

$$= \frac{1}{\kappa_2^{(q)}} \left[\frac{1+\frac{5!!}{4!}\gamma_4^{(q)} - \frac{7!!}{6!}\gamma_6^{(q)} + \frac{9!!}{8!}(\gamma_8^{(q)} + 35(\gamma_4^{(q)})^2) + \ldots}{1+\frac{3!!}{4!}\gamma_4^{(q)} - \frac{5!!}{6!}\gamma_6^{(q)} + \frac{7!!}{8!}(\gamma_8^{(q)} + 35(\gamma_4^{(q)})^2) + \ldots} \right]$$

Ratio of two infinite series

$$\approx \frac{1}{\kappa_2^{(q)}} \left[\frac{1 + \frac{5!!}{4!}\gamma_4^{(q)}}{1 + \frac{3!!}{4!}\gamma_4^{(q)}} \right]$$

Where do we truncate x^m, $m = 4$?

Source cumulants in terms of q cumulants

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Where do we truncate x^m, $m = 6$?

Source cumulants in terms of q cumulants

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Ratio of two infinite series

$$\approx \frac{1}{\kappa_2^{(q)}} \left[\frac{1+\frac{5!!}{4!}\gamma_4^{(q)} - \frac{7!!}{6!}\gamma_6^{(q)} + \frac{9!!}{8!}(\gamma_8^{(q)} + 35(\gamma_4^{(q)})^2)}{1+\frac{3!!}{4!}\gamma_4^{(q)} - \frac{5!!}{6!}\gamma_6^{(q)} + \frac{7!!}{8!}(\gamma_8^{(q)} + 35(\gamma_4^{(q)})^2)} \right]$$

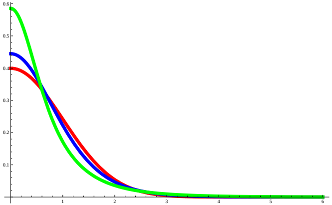
Where do we truncate x^m, $m = 8$?

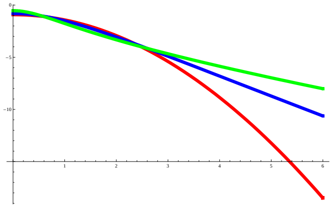
Test with a Toy Model

Test with Toy Model (Symmetrical Normal Inverse Gaussian):

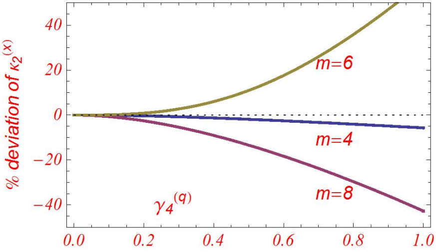
$$f(q|\gamma_2^{(q)},\gamma_4^{(q)}) = \frac{3e^{3/\gamma_4^{(q)}} K_1\left(3\gamma_4^{(q)}\sqrt{1+\frac{q^2}{3\gamma_4^{(q)}\gamma_2^{(q)}}}\right)}{\pi\sqrt{\left(\gamma_4^{(q)}\right)^2 q^2 + 3\gamma_4^{(q)}\gamma_2^{(q)}}}$$

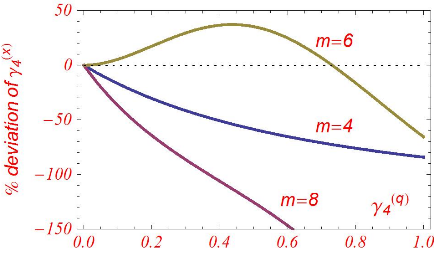
Adjustable kurtosis. Exact $\kappa_2^{(q)}, \kappa_4^{(q)}, \kappa_6^{(q)}, \kappa_2^{(x)}, \kappa_4^{(x)}, \kappa_6^{(x)}$.





Compare Toy model and Gram-Charlier





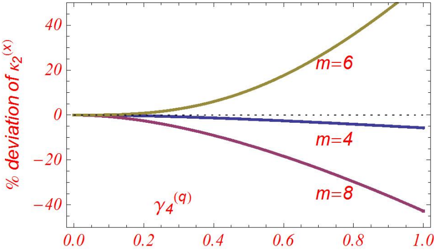
- Keep x^m terms in series.
- Percentage Deviation

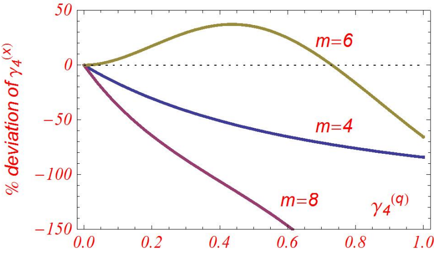
$$\frac{\kappa_j^{(x)}}{(\kappa_j^{(x)})_{SNIG}} - 1$$

- Against q-kurtosis

- Gaussian limit
- Asymptotic series becomes worse
- Unexpected disaster 50%-100% error

Compare Toy model and Gram-Charlier





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How do we fix this?

Cumulants of Toy model

$$\gamma_r^{(q)} \propto \left(\gamma_4^{(q)}\right)^{\frac{r}{2}-1}$$

Suggests truncation in γ_4^r. Assume $f(q)$ is n-divisible

$$q = q_1 + q_2 + \cdots + q_n$$

Product of n independent components

$$S(x) = S\left(\frac{x_i}{\sigma\sqrt{n}}\right)^n$$

Resulting cumulants

$$\gamma_r^{(q)} \propto \left(\frac{1}{n}\right)^{\frac{r}{2}-1}$$

Edgeworth series

Gram-Charlier series

$$S(x) = \exp\left[-\frac{x^2}{2} + \sum_{r=3}^{\infty} \frac{\gamma_r^{(q)} (ix)^r}{r!}\right]$$

Edgeworth series

$$S(x) = \exp\left[-\frac{x^2}{2} + \sum_{r=3}^{\infty} \frac{\gamma_r^{(q)} (ix)^r}{n^{r/2-1} r!}\right]$$

Central Limit Theorem

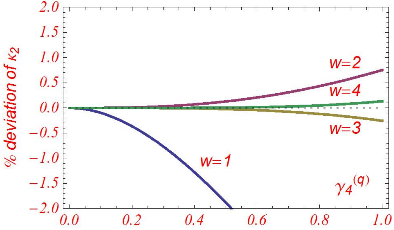
$$\lim_{n\to\infty} S(x) = \exp\left[-\frac{x^2}{2}\right]$$

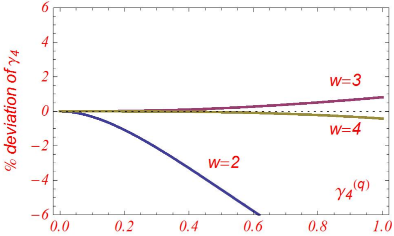
Edgeworth and Gram-Charlier

$$\frac{S(x)}{f(0)} = \begin{cases} \exp\left[-\frac{x^2}{2} + \sum_{r=3}^{\infty} \frac{\gamma_r^{(q)}(ix)^r}{r!}\right] = \sum x^m F_m & \text{Gram-Charlier} \\ \exp\left[-\frac{x^2}{2} + \sum_{r=3}^{\infty} \frac{\gamma_r^{(q)}(ix)^r}{n^{r/2-1}r!}\right] = \sum n^{-w} C_w & \text{Edgeworth} \end{cases}$$

Gram-Charlier	Edgeworth
Powers in x^m	Powers in n^{-w}
$\gamma_4 \frac{H_4(q)}{4!}$	$\frac{1}{n}\left[\gamma_4 \frac{H_4(q)}{4!}\right]$
$\gamma_6 \frac{H_6(q)}{6!}$	$\frac{1}{n^2}\left[35\gamma_4^2 \frac{H_8(q)}{8!} + \gamma_6 \frac{H_6(q)}{6!}\right]$
$\left[\gamma_8 + 35\gamma_4^2\right] \frac{H_8(q)}{8!}$	$\frac{1}{n^3}\left[\gamma_8 \frac{H_8(q)}{8!} + 210\gamma_6\gamma_4 \frac{H_{10}(q)}{10!} + 5775\gamma_4^3 \frac{H_{12}(q)}{12!}\right]$

Compare Edgeworth series and Toy model





- Keep terms up to order n^{-w}
- Percentage Deviation
  $$\frac{\kappa_j^{(x)}}{(\kappa_j^{(x)})_{SNIG}} - 1$$
- Against q-kurtosis

- Edgeworth accuracy 1%
- Improving approximation
- Independent of the value of n

Summary

- Edgeworth series can relate correlation cumulants to source cumulants.
- Gram-Charlier cannot.
- Gram-Charlier is a orthogonal polynomial expansion.
- Simple problem and text book method is surprisingly complex.
- Reordering gives a large decrease in error.
- Edgeworth groups terms according to their approach to normality.
- Why this works is still unclear.